TITLE: Shape Preserving Interpolation by Curves

# Shape preserving interpolation by curves

**T. N. T. Goodman**

*Department of Mathematics, University of Dundee, Dundee DD1 4HN.*
tgoodman@maths.dundee.ac.uk

**Abstract**

A survey is given of algorithms for passing a curve through data points so as to preserve the shape of the data.

## 1 Introduction

We consider the problem of passing a curve through a finite sequence of points. We want the curve to preserve in some sense the shape of the data, i.e. the shape of the curve gained by joining the data by straight line segments (which we call the 'piecewise linear interpolant'). We do not consider the important problems of *approximating* the data by a curve, or of shape-preserving interpolation by a surface. The short length of the paper forces it to be selective. So we concentrate on actual algorithms for solving the problem rather than related theory. Also we consider only algorithms where the curve is defined explicitly, not implicitly either as the zero set of a function or as the limit of a subdivision process (though there are, to our knowledge, extremely few such implicit shape-preserving schemes).

In Section 2, we consider planar curves given by a function $y = f(x)$, often rather misleadingly referred to as 'functional interpolation'. There are numerous such schemes, dating from 1966, with most of them prior to 1990. Our treatment is therefore very selective. Section 3 deals with parametrically defined planar curves, for which the schemes are fewer and more recent. Finally, in Section 4, we consider curves in three dimensions, often called 'space curves'. Here the work is much more limited, dating only from 1997.

We note that in shape-preserving interpolation, the map from the data to the function describing the curve must be non-linear. In what we call 'tension methods' the curve can be constructed by a linear scheme for any choice of certain 'tension parameters'. These parameters are then varied so as to 'pull' the curve towards the piecewise linear interpolant until the shape criteria are satisfied. Though there are a few variations on this theme, there is generally a clear distinction between tension methods and other schemes, which we shall term 'direct methods'.

## 2 Functional interpolation

Given data

$$(x_i, y_i) \in R^2, \quad i = 0, \ldots, N, \quad x_0 < x_1 < \cdots < x_N, \tag{2.1}$$

we consider a function $f : [x_0, x_N] \to R$ satisfying

$$f(x_i) = y_i, \quad i = 0, \ldots, N. \tag{2.2}$$

For some reasons, perhaps the physical situation which $f$ is intended to model, we may wish the graph of $f$ to inherit certain shape properties of the data. We now describe these and other properties which it may be desirable for $f$ to possess.

## 2.1 Desirable properties

**Monotonicity.** Here we require $f$ to be increasing (respectively decreasing) if $(y_i)$ is increasing (respectively decreasing). More generally we may require the scheme to be **'co-monotone'**, i.e. for $i = 0, \ldots, N-1$, $f$ is increasing (decreasing) on $[x_i, x_{i+1}]$ if $y_i \le y_{i+1}$ $(y_i \ge y_{i+1})$. Co-monotonicity has the consequence that the local extrema of $f$ occur exactly at the local extrema of $(y_i)$. Moreover if $y_i = y_{i+1}$, then $f$ is constant on $[x_i, x_{i+1}]$. These properties may be too restrictive and a weaker alternative is what we call **'local monotonicity'**: for $i = 1, \ldots, N-2$, $f$ is increasing on $[x_i, x_{i+1}]$ if $y_{i-1} \le y_i \le y_{i+1} \le y_{i+2}$ (and similarly for decreasing). Although this is not generally stated, it is also desirable that for $i = 0, \ldots, N-1$, $f$ has at most one local extremum on $(x_i, x_{i+1})$.

**Convexity.** Here we require $f$ to be convex (concave) if the piecewise linear interpolant is convex (concave). More generally we call the scheme **'co-convex'** if for $i = 1, \ldots, N-2$, $f$ is convex (concave) on $[x_i, x_{i+1}]$ if the piecewise linear interpolant is convex (concave) on $[x_{i-1}, x_{i+2}]$. It is also desirable in a co-convex scheme for $f$ to have at most one inflection in $(x_i, x_{i+1})$, $0 \le i \le N-1$.

**Smoothness.** By definition, the piecewise linear interpolant is shape-preserving, and so the problem is trivial unless we require $f$ to have greater smoothness than continuity, i.e. $C^k$ for $k \ge 1$. Since all the schemes use piecewise analytic functions, the $C^k$ condition needs to be checked only at a finite number of 'knots', which generally include the data points. We remark that smoothness and shape-preservation may not be compatible; e.g. if for $i = 0, \ldots, 4$, $x_i = i - 2$, $y_i = |x_i|$, and $f$ is convex on $[x_0, x_4]$, then $f(x) = |x|$, $-2 \le x \le 2$, and so is not $C^1$ at 0.

**Approximation order.** It is generally supposed that the data arise as values of some unknown 'smooth' function $g$, i.e. $y_i = g(x_i)$, $i = 0, \ldots, N$. Then we can consider how fast the interpolant $f$ converges to $g$ as we increase the density of data values $x_i$ in the fixed interval $[a, b]$. A scheme has approximation order $O(h^m)$ if $||f - g|| = O(h^m)$, where $h = \max\{x_{i+1} - x_i : i = 0, \ldots, N-1\}$ and the usual norm is $||F|| = \sup\{|F(x)| : a \le x \le b\}$.

**Locality.** In a 'global' scheme, the value $f(x)$, for any $x$, generally depends on all the data. In contrast, for a 'local' scheme, $f(x)$ depends on the data values $(x_i, y_i)$ only for $x_i$ 'near' $x$. There may be advantages in local schemes, e.g. when data are modified or inserted.

**Fairness.** It is often desirable that the curve is 'fair', i.e. pleasing to the eye, see Section 3.

Other desirable properties are invariance under scaling or reflection in $x$ or $y$, and stability, i.e. small changes in the data produce small changes in $f$. There may also be other constraints on $f$, e.g. $f \geq 0$ when $y_i \geq 0$, $i = 0, \ldots, N$.

## 2.2 Tension methods

Many tension methods are a modification of cubic spline interpolation, which we now describe. Given data (2.1), there is a unique function $f$ satisfying (2.2), where $f$ is $C^2$, is a cubic polynomial on $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, and satisfies suitable boundary conditions at $x_0$ and $x_N$. The function $f$ minimises $\int_{x_0}^{x_N} (g'')^2$ over a suitable class of functions and this energy minimisation property is generally considered to give a fair curve. Determining $f$ requires solving a global, strictly diagonally dominant tridiagonal system of linear equations.

Since cubic spline interpolation is not shape-preserving, in 1966 Schweikert [67] modified the scheme by replacing cubic polynomials on each interval $[x_i, x_{i+1}]$ by solutions of

$$f^{(4)} - \lambda_i f'' = 0,$$

where $\lambda_i \geq 0$. When $\lambda_i = 0$, $f$ will reduce to a cubic, while as $\lambda_i \to \infty$, $f$ approaches a linear polynomial. Thus $\lambda_i$ acts as a tension parameter and by making appropriate choices of $\lambda_i$ large enough the function will preserve monotonicity and/or convexity globally or locally.

Many papers have been written on Schweikert's tension splines giving, for example, ways of choosing the values of the tension parameters, e.g. [68,57,46,60]. However the fact that the method uses exponential functions can be seen as a drawback. An alternative was introduced by Nielson in 1974 [55] by adjusting the minimisation property of cubic splines to a minimisation problem involving also the first derivative. The resulting function, called a $\nu$-spline, is also cubic on each interval $[x_i, x_{i+1}]$ but only $C^1$. However the form of the $C^1$ continuity gives extra 'smoothness' for parametrically defined curves and so we discuss $\nu$-splines further in Section 3. By generalising the minimisation problem still further one can gain a $C^1$ piecewise cubic interpolant with further parameters for gaining shape properties [22].

The idea of using rational functions in tension methods was introduced by Späth [69], also in 1974, and put in a general setting of tension methods in [57]. From 1982–1988, Gregory and/or Delbourgo produced a series of algorithms using rational functions, e.g. [19,36,20,21,18]. We illustrate the ideas with an algorithm from [37]. Here $f$ is $C^2$ and on each interval $[x_i, x_{i+1}]$ it has the form, for some $a$, $b$, $c$, $d$,

$$f(t) = \frac{a + bt + ct^2 + dt^3}{1 + \lambda_i t(1-t)}, \quad t = \frac{x - x_i}{x_{i+1} - x_i}.$$

For $\lambda_i > -1$, $i = 0, \ldots, N-1$, $f$ can be determined as the solution of a strictly diagonally dominant tridiagonal linear system (and hence the scheme is global). When all $\lambda_i = 0$, $f$ reduces to the usual cubic spline interpolant, while as $\lambda_i \to \infty$, $f$ converges uniformly to the linear interpolant on $[x_i, x_{i+1}]$. In general the approximation order is $O(h^2)$ for

data from a $C^4$ function. In the special case of monotone data, choosing

$$\lambda_i = \mu_i + (f'(x_i) + f'(x_{i+1}))\frac{x_{i+1} - x_i}{y_{i+1} - y_i}, \quad \mu_i \geq -3, \quad i = 0, \ldots, N-1,$$

ensures that $f$ is correspondingly monotone, and for the choice $\mu_i = -2$, $f$ reduces to a rational quadratic which gives optimal approximation order $O(h^4)$. Similarly for convex data, $f$ is also convex provided that each $\lambda_i$ satisfies an inequality involving $f'(x_i)$, $f'(x_{i+1})$, and choosing $\lambda_i$ appropriately (which requires solving a non-linear equation) further ensures approximation order $O(h^4)$.

There are some more recent methods involving rationals, e.g. [58].

The idea of using variable degree to preserve shape was introduced by McAllister, Passow and Roulier in 1977 [47,56]. They produce monotone, convex schemes of arbitrarily high smoothness by constructing a shape-preserving piecewise linear interpolant $l$ with one knot between any two data points (and no knots at the data points) and then defining the final interpolant on each interval $[x_i, x_{i+1}]$ as the Bernstein polynomial of $l$ of some degree $m_i$. The idea was extended from 1986 by Costantini [8–10]. For $k \geq 1$, $m_i \geq 2k+1$, $i = 0, \ldots, N-1$, he constructs a shape-preserving piecewise linear interpolant $l$ with knots at $x_i + k(x_{i+1} - x_i)/m_i$ and $x_{i+1} - k(x_{i+1} - x_i)/m_i$, $i = 0, \ldots, N-1$. The final interpolant $f$ coincides on each interval $[x_i, x_{i+1}]$ with the Bernstein polynomial of $l$ of degree $m_i$ and is hence $C^k$ (with $f^{(j)}(x_i) = 0$, $j = 2, \ldots, k$). In [10] there is a co-monotone, co-convex scheme in which the degrees $m_i$ can either be chosen a priori or computed automatically according to the data.

The above schemes using variable degree are not strictly tension schemes in our sense but in 1990, Kaklis and Pandelis [40] introduced a tension method by using the above form for $k = 1$, i.e. on each interval $[x_i, x_{i+1}]$ it has the form:

$$f(t) = f(x_i)(1-t) + f(x_{i+1})t + c_i t(1-t)^{m_i} + d_i t^{m_i}(1-t), \quad t = \frac{x - x_i}{x_{i+1} - x_i}.$$

Here $m_i \geq 2$ is an integer and for each choice of $m_0, \ldots, m_{N-1}$, the numbers $c_i$, $d_i$ are chosen so that $f$ is $C^2$, which requires the solution of a strictly diagonally dominant tridiagonal linear system. When all $m_i = 2$, this reduces to the usual cubic spline interpolant, while as $m_i \to \infty$, $f$ converges uniformly to the linear interpolant on $[x_i, x_{i+1}]$ with order $O(m_i^{-1})$ (or $O(m_i^{-2})$ if $m_{i-1}$, $m_{i+1}$ remain bounded). For further discussion of variable degree shape-preserving functional interpolation, see [11].

Our final type of tension method was introduced by Manni [50] in 1996. The general idea is to define $f$ on $[x_i, x_{i+1}]$ as

$$f(x) = p_i(q_i^{-1}(x)),$$

where $p_i$, $q_i$ are cubic polynomials on $[x_i, x_{i+1}]$ and $q_i$ is strictly increasing from $[x_i, x_{i+1}]$ onto itself, so that the inverse $q_i^{-1}$ is well-defined on $[x_i, x_{i+1}]$. For $f'(x_i) = d_i$, $i = 0, \ldots, N$, we require

$$p_i'(x_i) = \lambda_i d_i, \quad q_i'(x_i) = \lambda_i, \quad p_i'(x_{i+1}) = \mu_i d_{i+1}, \quad q_i'(x_{i+1}) = \mu_i,$$

for parameters $\lambda_i > 0$, $\mu_i > 0$. For $\lambda_i = \mu_i = 1$, we have $q_i(x) = x$ and $f$ reduces to a cubic on $[x_i, x_{i+1}]$, while for $\lambda_i = \mu_i = 0$, $f$ becomes linear on $[x_i, x_{i+1}]$.

In [50], the values $d_0, \dots, d_N$ are assumed known (or estimated from the data values) and the scheme is local $C^1$, gives necessary and sufficient conditions for the values of the parameters $\lambda_i$, $\mu_i$ for co-monotonicity, and has approximation order $O(h^2)$ when $g$ is $C^2$ and generally $O(h^4)$ when $g$ is $C^4$.

Manni and co-workers have written a series of papers using the same idea, [51,53,54]. For example in [45], the values $d_i$ are not assumed given but are chosen to ensure that the function is $C^2$, thus providing a locally monotone, co-convex global scheme which generalises usual cubic spline interpolation; while in [52] two further knots are inserted in each interval $[x_i, x_{i+1}]$ to produce a $C^2$, locally monotone, co-convex local scheme which interpolates values of $f^{(j)}(x_i)$, $j = 1, 2$, $i = 0, \dots, N$.

### 2.3 Direct methods

In 1967, Young [71] considered shape-preserving interpolation by polynomials and a number of papers have appeared since on this topic, e.g. [59] gives a constructive proof of the existence of a co-monotone interpolant with an upper bound on the degree required. However for a practical algorithm, using a piecewise polynomial offers much more flexibility than a single polynomial. Numerous papers have been written using such polynomial splines and we mention briefly only a few.

By inserting extra knots between data points, a convexity preserving scheme with $C^2$ cubics was given by de Boor [4, p.303], and co-monotone, co-convex schemes with $C^1$ quadratics in [48,49,66]. $C^1$ cubic splines with knots at the data points are used for co-monotonicity in [25,5,24,70], (the last of these using a variational approach), and for both co-monotonicity and co-convexity in [16,17]. We also recall the methods using spline functions of variable degree with knots between the data points to obtain interpolants with arbitrarily high smoothness which were discussed under tension methods.

Finally we note that following the paper [62] which was as early as 1973, Schaback [63] gives a $C^2$ co-monotone, co-convex scheme which uses a cubic polynomial on any interval $[x_i, x_{i+1}]$ where an inflection is needed, and on other intervals employs a rational function of form quadratic/linear.

## 3 Planar curves

Given data

$$I_i \in R^2, \quad i = 0, \dots, N,$$

we consider a curve $r : [a, b] \to R^2$ satisfying

$$r(t_i) = I_i, \quad i = 0, \dots, N, \tag{3.1}$$

for values $a = t_0 < t_1 < \cdots < t_N = b$. For a closed curve the situation is extended periodically so that

$$I_{i+N} = I_0, \quad t_{i+N} = t_i, \quad i \in Z, \quad r(t + b - a) = r(t), \quad t \in R.$$

### 3.1 Desirable properties

**Shape.** For this case it is not usually relevant to consider preservation of monotonicity. We say a scheme is **'co-convex'** if the curve $r$ has the minimum number of inflections

consistent with the data. In practice, schemes satisfy the somewhat stronger condition that for any $0 \le i \le j-2 \le N-2$, $r$ is positively (negatively) locally convex on $[t_{i+1}, t_{j-1}]$ if the polygonal arc joining $I_i, \ldots, I_j$ is positively (negatively) locally convex. For more details on this and other desirable properties, see [29].

**Smoothness.** We shall call the interpolating curve $C^k$ for $k \ge 0$ if the function $r$ is $C^k$. A $C^0$ curve $r$ we shall call $G^1$ if the unit tangent vector is continuous, and $G^2$ if, in addition, the curvature is continuous. A $C^k$ curve $r$ is $G^k$, $k = 1, 2$, provided that the parameterisation is regular, i.e. $r'(t) \ne (0,0)$, which is generally desirable. It is usually sufficient to have $G^k$, rather than $C^k$, continuity if only the appearance of the curve is important and the choice of parameter $t$ is not significant.

**Fairness.** Planar curves often arise in computer-aided design where it may be particularly important that the curve is pleasing to the eye. Though this is subjective, various criteria have been suggested to be relevant, such as magnitude, rate of change or monotonicity of the curvature. Some schemes include 'shape parameters' which can be manipulated by the designer to modify the shape of the curve.

**Approximation order** is not important in the context of design when the data are not considered to be taken from some unknown curve. Approximation order is related to reproduction of polynomial curves, and a related property for planar curves is reproduction of arcs of circles (or more generally conics); this cannot be done exactly by polynomials but it can be achieved by using rationals.

Locality and other desirable properties are similar to the functional case as described in Section 2.1, though it is generally more appropriate that the invariance is under a rotation and the same scaling in both $x$ and $y$.

### 3.2 Tension methods

In Section 2.2 we mentioned Nielson's $\nu$-splines [55]. Applying this scheme for both components of $r$ gives a function $r$ which is cubic on each interval $[t_i, t_{i+1}]$, is $C^1$ and satisfies

$$r''(t_i^+) = r''(t_i^-) + \nu_i r'(t_i), \quad i = 1, \ldots, N-1,$$

where $\nu_i \ge 0$. This condition is sufficient for $G^2$ continuity of $r$ (assuming regular parameterisation). When all $\nu_i = 0$, $r$ will reduce to the usual $C^2$ cubic spline interpolant. As $\nu_i \to \infty$, the curve is 'pulled tight' at $I_i$ and as $\nu_i$, $\nu_{i+1} \to \infty$, it approaches the linear interpolant on $[t_i, t_{i+1}]$.

The scheme in [37] by Gregory which was mentioned in Section 2.2 was adapted to the planar case in [38]. Other schemes using rationals were proposed by Clements in [6,7], where $r$ is a $C^2$ curve which on each interval $[t_i, t_{i+1}]$ has the form, for some $a$, $b$, $c$, $d \in R^2$,

$$r(t) = \frac{a(1-s)^3}{w_i s + 1} + b(1-s) + cs + \frac{ds^3}{w_i(1-s)+1}, \quad s = \frac{t - t_i}{t_{i+1} - t_i},$$

where $w_i \ge 0$ are the tension parameters.

The variable degree tension method of [40], also mentioned in Section 2.2, was adapted to the planar case in [41], and extended in [27] to allow the designer to obtain a 'fair' curve by minimising the number of changes in the monotonicity of the curvature.

### 3.3 Direct methods

The papers [34,35,28,23] give local, $G^2$ co-convex schemes, e.g. in [28], a rational cubic/cubic is used on each interval $[t_i, t_{i+1}]$ and the tangent vectors and curvatures are stipulated by the algorithm to ensure that the convexity conditions are satisfied and circular arcs are reproduced, with the possibility of modifying the tangent vectors and curvatures further as shape parameters.

Following an earlier scheme in [64], Schaback in [65] gives a global $G^2$ co-convex scheme which uses a cubic polynomial on any interval $[t_i, t_{i+1}]$ where an inflection is needed, and on other intervals employs quadratic polynomials.

Sapidis and Kaklis [61] give a $G^2$ co-convex scheme by interpolating by a piecewise quintic curve tangent directions and curvatures gained by their tension method [41].

In [1] a local, co-convex $G^2$ scheme is given which uses polynomials of degree six and which attempts to obtain a fair curve by imposing conditions on the curvature to minimise measures of fairness. Finally we note that in [12] Costantini gives an abstract theory and general purpose code.

## 4 Space curves

Given data

$$I_i \in R^3, \quad i = 0, \ldots, N,$$

we consider a curve $r : [a, b] \to R^3$ satisfying condition (3.1) as before.

### 4.1 Desirable properties

What is meant by 'shape-preserving' is not so clear for space curves as for the planar case. Criteria were introduced by Kaklis and Karavelas [39] and extended by Ong and the author in [31]. We shall sketch these below. They are discussed in further detail in [30], where some further extensions are suggested. We write, for appropriate indices $i$:

$$L_i = I_{i+1} - I_i, \quad \Delta_i = \det[L_{i-1}, L_i, L_{i+1}], \quad N_i = L_{i-1} \times L_i.$$

**Torsion.** We ensure that the curve is 'twisting' in the same manner as the piecewise linear interpolant by requiring that if $\Delta_i \neq 0$, then the torsion of $r$ has the same sign as $\Delta_i$ on $(t_i, t_{i+1})$.

**Convexity.** Let

$$K(t) = r'(t) \times r''(t), \quad a \leq t \leq b.$$

We require that for $1 \leq i \leq N-1$, $K(t_i).N_i > 0$, which means that the projection of the curve $r$ onto the plane of $I_{i-1}$, $I_i$, $I_{i+1}$, has the same sign of local convexity at $I_i$ as the polygonal arc $I_{i-1}I_iI_{i+1}$. Moreover if $N_i.N_{i+1} > 0$, we require

$$K(t).N_j > 0, \quad j = i, i+1, \quad t_i \leq t \leq t_{i+1},$$

which implies that the curve $r$ has the same sign of local convexity on $[t_i, t_{i+1}]$ when projected in any direction $\lambda N_i + (1 - \lambda)N_{i+1}$ for $0 \leq \lambda \leq 1$. Finally we require that if $N_i.N_{i+1} < 0$, then for $j = i,\ i+1,\ K(t).N_j$ has exactly one sign change in $[t_i, t_{i+1}]$, which imples that each of the above projections of $r$ have just one inflection.

**Smoothness.** This is as for planar curves, except that we call the curve $G^3$ if it is $G^2$ and, in addition, the torsion is continuous. Other desirable properties are similar to the planar case.

## 4.2 Tension methods

Although interpolation by space curves with a special shape is considered in [44], the first specific shape-preserving interpolation scheme by space curves was due to Kaklis and Karavelas [39], who adapted the variable degree tension method of [40] to give a $C^2$ method which was also $G^3$, but at the expense of zero torsion at the data points. In [42] the same authors adapted Nielson's $\nu$-splines to the three dimensional case to give a curve which is $C^1$ and $G^2$. The paper [14] also uses variable degree for tension parameters but gives a $C^3$ scheme in which the limiting curve as the tension goes to infinity is not the piecewise linear interpolant but the shape-preserving interpolant given by either of the above two schemes. In [15] a $C^3$ scheme is also given but here the components of $r$ on each interval $[t_i, t_{i+1}]$ lie in the linear span of the functions

$$(1-u), u, (1-u)^{m_i}, (1-u)^{m_i-1}u, (1-u)u^{m_{i+1}-1}, u^{m_{i+1}}, \quad u = \frac{t-t_i}{t_{i+1}-t_i}.$$

When $m_i = m_{i+1} = 5$, this reduces to a quintic polynomial. As $m_i$, $m_{i+1} \to \infty$, it tends to a linear polynomial and then the curve $r$ approaches the piecewise linear interpolant on $[t_i, t_{i+1}]$.

The paper [26] also uses variable degree splines with degree on each interval at least five, and the curve $r$ also converges to the piecewise linear interpolant as the degrees go to infinity. However here the curve is $C^4$, which the authors feel may give extra fairness to the curve due, for example, to lowering the maximum absolute value of the curvature. Variable degree polynomial splines are also used in [13].

## 4.3 Direct methods

Following an earlier scheme in [31], Ong and the author gave a local $G^2$ scheme in [32] which employed a rational cubic/cubic between data points, extending the ideas of the planar scheme in [28]. This was further extended to a local $G^3$ scheme using a rational quartic/quartic in [43]. In [33], the degrees of freedom inherent in the scheme in [32] were used to optimise a fairness measure. Finally we mention the papers [2,3] which give local $G^3$ schemes using a piecewise polynomial of degree six, also allowing optimisation of a fairness measure.

It will be noted that many of the above papers are extremely recent and it is hoped that the unavoidable lack of detail here will serve to tantalise readers to discover for themselves more of this rapidly developing field.

## Bibliography

1. S. Asaturyan, P. Costantini and C. Manni, $G^2$ shape preserving parametric planar curve interpolation, in Creating Fair and Shape-Preserving Curves and Surfaces, H. Nowacki, P. D. Kaklis (eds.), B. G. Teubner, Stuttgart (1998), 89–98.
2. S. Asaturyan, P. Costantini and C. Manni, Shape-preserving interpolating curves in $R^3$: a local approach, in Creating Fair and Shape-Preserving Curves and Surfaces, H. Nowacki, P. D. Kaklis (eds.), B. G. Teubner, Stuttgart (1998), 99–108.
3. S. Asaturyan, P. Costantini and C. Manni, Local shape-preserving interpolation by space curves, IMA J. Numer. Anal. **21** (2001), 301–325.
4. C. de Boor, A Practical Guide to Splines, Springer, New York (1978).
5. J. Butland, A method of interpolating reasonable-shaped curves through any data, Proc. Computer Graphics 80, Online Publ. Ltd., Northwood Hills, Middlesex, U.K. (1980), 409–422.
6. J. C. Clements, Convexity-preserving piecewise rational cubic interpolation, SIAM J. Numer. Anal. **27** (1990), 1016–1023.
7. J. C. Clements, A convexity-preserving $C^2$ parametric rational cubic interpolant, Numer. Math. **63** (1992), 165–171.
8. P. Costantini, On monotone and convex spline interpolation, Math. Comp. **46** (1986), 203–214.
9. P. Costantini, Co-monotone interpolating splines of arbitrary degree - a local approach, SIAM J. Sci. Stat. Comput. **8** (1987), 1026–1034.
10. P. Costantini, An algorithm for computing shape-preserving interpolating splines of arbitrary degree, J. Comput. Appl. Math. **22** (1988), 89–136.
11. P. Costantini, Abstract schemes for functional shape-preserving interpolation, in Advanced Course on Fairshape, J. Hoschek, P. Kaklis (eds.), B. G. Teubner, Stuttgart (1996), 185–199.
12. P. Costantini, Boundary-valued shape-preserving interpolating splines, ACM Trans. on Math. Software **23** (1997), 229–251.
13. P. Costantini, Curve and surface construction using variable degree polynomial splines, Computer Aided Geometric Design **17** (2000), 419–446.
14. P. Costantini, T. N. T. Goodman and C. Manni, Constructing $C^3$ shape preserving interpolating space curves, Advances Comp. Math. **14** (2001), 103–127.
15. P. Costantini and C. Manni, Shape-preserving $C^3$ interpolation: the curve case, to appear.
16. P. Costantini and R. Morandi, Monotone and convex cubic spline interpolation, Calcolo **21** (1984), 281–294.
17. P. Costantini and R. Morandi, An algorithm for computing shape-preserving cubic spline interpolation to data, Calcolo **21** (1984), 295–305.
18. R. Delbourgo, Shape preserving interpolation to convex data by rational functions with quadratic numerator and linear denominator, IMA J. Numer. Anal. **9** (1989), 123–136.
19. R. Delbourgo and J. A. Gregory, $C^2$ rational quadratic spline interpolation to mono-

tonic data, IMA J. Numer. Anal. **3** (1983), 141–152.

20. R. Delbourgo and J. A. Gregory, The determination of derivative parameters for a monotonic rational quadratic interpolant, IMA J. Numer. Anal. **5** (1985), 397–406.
21. R. Delbourgo and J. A. Gregory, Shape preserving piecewise rational interpolation, SIAM J. Sci. Stat. Comput. **6** (1985), 967–976.
22. T. A. Foley, A shape preserving interpolant with tension controls, Computer Aided Geometric Design **5** (1988).
23. T. A. Foley, T. N. T. Goodman and K. Unsworth, An algorithm for shape-preserving parametric interpolating curves with $G^2$ continuity, in Mathematical Methods in CAGD, T. Lyche, L. L. Schumaker (eds.), Academic Press, Boston (1989), 249–259.
24. F. N. Fritsch and J. Butland, A method for constructing local monotone piecewise cubic interpolants, SIAM J. Sci. Stat. Comput. **5** (1984), 300–304.
25. F. N. Fritsch and R. E. Carlson, Monotone piecewise cubic interpolation, SIAM J. Numer. Anal. **17** (1980), 238–246.
26. N. C. Gabrielides and P. D. Kaklis, $C^4$ interpolatory shape-preserving polynomial splines of variable degree, Computing **65** (2001), to appear.
27. A. Ginnis, P. Kaklis and N. S. Sapidis, Polynomial splines of non-uniform degree: controlling convexity and fairness, in Designing Fair Curves and Surfaces, N. S. Sapidis (ed.), SIAM Series on Geometric Design, Philadelphia (1994), Part 3, Chapter 10.
28. T. N. T. Goodman, Shape preserving interpolation by parametric rational cubic splines, in Numerical Mathematics Singapore 1988, R. P. Agarwal, Y. M. Chow, S. J. Wilson (eds.), International Series of Numerical Mathematics Vol. 86, Birkhauser Verlag, Basel (1988), 149–158.
29. T. N. T. Goodman, Shape preserving interpolation by planar curves, in Advanced Course on Fairshape, J. Hoschek, P. Kaklis (eds.), B. G. Teubner, Stuttgart (1996), 29–38.
30. T. N. T. Goodman and B. H. Ong, Shape preserving interpolation by curves in three dimensions, in Advanced Course on Fairshape, J. Hoschek, P. Kaklis (eds.), B. G. Teubner, Stuttgart (1996), 39–48.
31. T. N. T. Goodman and B. H. Ong, Shape preserving interpolation by space curves, Computer Aided Geometric Design **15** (1997), 1–17.
32. T. N. T. Goodman and B. H. Ong, Shape preserving interpolation by $G^2$ curves in three dimensions, in Curves and Surfaces with Applications in CAGD, A. LeMehauté, C. Rabut, L. L. Schumaker (eds.), Vanderbilt Univ. Press, Nashville (1997), 151–158.
33. T. N. T. Goodman, B. H. Ong and M. L. Sampoli, Automatic interpolation by fair, shape preserving, $G^2$ space curves, Computer-aided Design **30** (1998), 813–822.
34. T. N. T. Goodman and K. Unsworth, Shape preserving interpolation by parametrically defined curves, SIAM J. Numer. Anal. **25** (1988), 1453–1465.
35. T. N. T. Goodman and K. Unsworth, Shape preserving interpolation by curvature continuous parametric curves, Computer Aided Geometric Design **5** (1988), 323–

340.

36. J. A. Gregory, Shape preserving rational spline interpolation, in Rational Approximation and Interpolation, Graves-Morris, Saff and Varga (eds.), Springer-Verlag (1984), 431–441.
37. J. A. Gregory, Shape preserving spline interpolation, Computer-aided Design **18** (1986), 53–58.
38. J. A. Gregory and M. Sarfraz, A rational cubic spline with tension, Computer Aided Geometric Design **7** (1990), 1–13.
39. P. D. Kaklis and M. I. Karavelas, Shape preserving interpolation in $R^3$, IMA J. Numer. Anal. **17** (1997), 373–419.
40. P. D. Kaklis and D. G. Pandelis, Convexity-preserving polynomial splines of non-uniform degree, IMA J. Numer. Anal. **10** (1990), 223–234.
41. P. D. Kaklis and N. S. Sapidis, Convexity-preserving interpolating parametric splines of non-uniform polynomial degree, Computer Aided Geometric Design **12** (1995), 1–26.
42. M. I. Karavelas and P. D. Kaklis, Spatial shape-preserving interpolation using $\nu$-splines, Numerical Algorithms **23** (2000), 217–250.
43. V. P. Kong and B. H. Ong, Shape preserving interpolation using Frenet frame continuous curves of order 3, to appear.
44. C. Labenski and B. Piper, Coils, Computer Aided Geometric Design **20** (1996), 1–29.
45. P. Lamberti and C. Manni, Shape preserving $C^2$ functional interpolation via parametric cubics, Numerical Algorithms, to appear.
46. R. W. Lynch, A method for choosing a tension factor for spline under tension interpolation, M.Sc. Thesis, Univ. of Texas at Austin (1982).
47. D. F. McAllister, E. Passow and J. A. Roulier, Algorithms for computing shape preserving spline interpolation to data, Math. Comp. **31** (1977), 717–725.
48. D. F. McAllister and J. A. Roulier, An algorithm for computing a shape preserving osculating quadratic spline, ACM Trans. Math. Software **7** (1981), 331–347.
49. D. F. McAllister and J. A. Roulier, Algorithm 574. Shape preserving osculating quadratic splines, ACM Trans. Math. Software **7** (1981), 384–386.
50. C. Manni, $C^1$ comonotone Hermite interpolation via parametric cubics, J. Comp. Appl. Math. **69** (1996), 143–157.
51. C. Manni, Parametric shape-preserving Hermite interpolation by piecewise quadratics, in Advanced Topics in Multivariate Approximation, F. Fontanella, K. Jetter, P. J. Laurent (eds.), World Scientific (1996), 211–228.
52. C. Manni, On shape preserving $C^2$ Hermite interpolation, BIT **14** (2001), 127–148.
53. C. Manni and P. Sablonnière, Monotone interpolation of order 3 by $C^2$ cubic splines, IMA J. Numer. Anal. **17** (1997), 305–320.
54. C. Manni and M. L. Sampoli, Comonotone parametric Hermite interpolation, in Mathematical Methods for Curves and Surfaces II, M. Daehlen, T. Lyche, L. L. Schumaker (eds.), Vanderbilt Univ. Press, Nashville (1998), 343–350.

55. G. M. Nielson, Some piecewise polynomial alternatives to splines under tension, in Computer Aided Geometric Design, R. E. Barnhill, R. F. Riesenfeld (eds.), Academic Press (1974), 209–235.
56. E. Passow and J. A. Roulier, Monotone and convex interpolation, SIAM J. Numer. Anal. **14** (1977), 904–909.
57. S. Pruess, Properties of splines in tension, J. Approx. Theory **17** (1976), 86–96.
58. R. Qu and M. Sarfraz, Efficient method for curve interpolation with monotonicity preservation and shape control, Neural, Parallel and Scientific Computations **5** (1997), 275–288.
59. L. Raymon, Piecewise monotone interpolation in polynomial type, SIAM J. Math. Anal. **12** (1981), 110–114.
60. N. S. Sapidis, P. D. Kaklis and T. A. Loukakis, A method for computing the tension parameters in convexity preserving spline-in-tension interpolation, Numer. Math. **54** (1988), 179–192.
61. N. S. Sapidis and P. D. Kaklis, A hybrid method for shape-preserving interpolation with curvature-continuous quintic splines, Computing Suppl. **10** (1995), 285–301.
62. R. Schaback, Spezielle rationale Splinefunktionen, J. Approx. Theory **7** (1973), 281–292.
63. R. Schaback, Adaptive rational splines, NAM-Bericht Nr. 60, Universität Göttingen (1988).
64. R. Schaback, Interpolation in $R^2$ by piecewise quadratic visually $C^2$ Bézier polynomials, Computer Aided Geometric Design **6** (1989), 219–233.
65. R. Schaback, On global $GC^2$ convexity preserving interpolation of planar curves by piecewise Bézier polynomials, in Mathematical Methods in CAGD, T. Lyche, L. L. Schumaker (eds.), Academic Press, Boston (1989), 539–548.
66. L. L. Schumaker, On shape preserving quadratic spline interpolation, SIAM J. Numer. Anal. **20** (1983), 854–864.
67. D. G. Schweikert, An interpolation curve using a spline in tension, J. Math. Phys. **45** (1966), 312–317.
68. H. Späth, Exponential spline interpolation, Computing **4** (1969), 225–233.
69. H. Späth, Spline algorithms for curves and surfaces, Utilitas Mathematica Pub. Inc., Winnipeg (1974).
70. F. I. Utreras and V. Celis, Piecewise cubic monotone interpolation: a variational approach, Departamento de Matematicas, Universidad de Chile, Tech. Report MA-83-B-281 (1983).
71. S. W. Young, Piecewise monotone polynomial interpolation, Bull. Amer. Math. Soc. **73** (1967), 642–643.